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The Harsanyi set for cooperative TU-games [1]

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Abstract

A cooperative game with transferable utilities, or simply a TU-game, describes a situation in which players can obtain certain payoffs by cooperation. A solution mapping for these games is a mapping which assigns to every game a set of payoff distributions over the players in the game. Well-known solution mappings are the Core and the Weber set. In this paper we consider the mapping assigning to every game the Harsanyi set being the set of payoff vectors obtained by all possible distributions of the Harsanyi dividends of a coalition amongst its members. We discuss the structure and properties of this mapping and show how the Harsanyi set is related to the Core and Weber set. We also characterize the Harsanyi mapping as the unique mapping satisfying a set of six axioms. Finally we discuss some properties of the Harsanyi Imputation set, being the individally rational subset of the Harsanyi set.

Key words: TU-games, Core, Harsanyi Set, Weber Set, Shapley value, Selectope.

JEL-code: C71.

1 Introduction

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utilities*, or simply a TU-game, being a pair (N, v), where $N = \{1, \ldots, n\}$ is a finite set of n players and $v: 2^N \to \mathbb{R}$ is a *characteristic function* on N such that $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S)$ is the worth of coalition S, i.e. the members of coalition S can obtain a total payoff of $v(S)$ by agreeing to cooperate.

A *solution vector* of an n-person TU-game is an n-dimensional vector giving a payoff to any player $i \in N$. A *solution function* is a function f that assigns a solution vector $f(v) \in \mathbb{R}^n$ to any game (N, v). A soluton function f efficiently distributes the worth of the grand coalition if for any game the total payoff it assigns to the players is equal to the worth $v(N)$ of the 'grand coalition', i.e. $\sum_{i \in N} f_i(v) = v(N)$ for any game (N, v). An example of an efficient solution function is the *Shapley value*, see Shapley [10], being the average of the so-called *marginal value vectors*.

A *solution mapping* is a mapping M that assigns to every game (N, v) a set of solution vectors in $\mathbb{R}^n$. Well-known solution mappings are the Core and the Weber set. The *Core* of a game, introduced in game theory by Gillies [2] is the set of all undominated payoff vectors, i.e. at any payoff vector in the Core each coalition gets at least its own value. The *Weber set*, see Weber [18], is defined as the convex hull of the marginal value vectors and contains the Core and the Shapley value. For the subset of so-called *convex* games the Core and Weber set coincide.

In this paper we discuss extensively the properties of the solution mapping assigning to each TU-game the set of all possible distributions of the so-called *Harsanyi dividends*, see Harsanyi [4], [5], of the coalitions amongst their members. For this reason we call this mapping the Harsanyi mapping and the set of solutions assigned to a game the *Harsanyi set*. In the seventies this set and related concepts have been introduced independently by Vasil'ev [12], [13] (both in Russian) and [14], and by Hammer, Peled and Sorensen [3]. Recently this set of solutions has been discussed by Derks, Haller and Peters [1] as the so-called *selectope*, being the convex hull of all so-called *selector values*. In Derks et al this selectope or Harsanyi set is studied from a set-theoretic point of view and from a value-theoretic point of view. In this paper we consider both the Harsanyi set and the *Harsanyi Imputation set*, being the subset of all individually rational payoff vectors in the Harsanyi set.

The purpose of this paper is twofold. From a historical viewpoint we recall several results already stated in the 1975, 1978 and 1981 papers of Vasil'ev. It shows that some of the results given in Derks et al [1] can be found already in these papers. As mentioned by Derks et al, some of these results can also be found in Hammer et al [3]. In particular, these results show that the Harsanyi set has a Core-type structure in the sense that the Harsanyi set of a game (N, v) is equal to the Core of a well-defined corresponding game, the so-called Harsanyi mingame of (N, v).

Besides these historical notes the paper also contains several new results. First, for any coalition $S \subseteq N$, we provide sharp lower and upper bounds for the minimum, respectively maximum payoff that coalition S can obtain at a payoff vector in the Harsanyi set. It shows that a coalition S obtains at least the worth of S in the Harsanyi mingame and at most the worth of S in the so-called Harsanyi maxgame. Second, we discuss extensively the relation between the Harsanyi set and the Weber set. In particular we characterize the Weber set as a subset of the Harsanyi set, in the sense that we show that we obtain the Weber set when we put certain restrictions on the possible distributions of the *Harsanyi dividends* of the coalitions amongst their members. Recall that the Harsanyi set is obtained when we allow for all possible distributions. To obtain this characterization we had to prove several results which are interesting in itself. The main result in this respect characterizes the extreme points of a certain polyhedron in the vector space Q, where Q is the set all possible weight systems, such that each weight system in Q gives a distribution of the worth of the grand coalition by assigning to each player the weighted sum of her marginal values. Third, we prove a result already mentioned in Vasil'ev [14], namely that the Harsanyi mapping is the only solution mapping satisfying a set of six axioms, namely convexity, efficiency, dummy player property, sign preserving property, individually consistency and disjoint additivity. Finally, we give several properties of the Harsanyi Imputation set. In particular we prove that this set is external stable in the sense of Von Neumann-Morgenstern.

This paper is organized as follows. In Section 2 we recall some preliminaries and basic concepts, as well as the solution concepts of the Shapley value, Core and Weber set. In Section 3 we define the Harsanyi set, recall some historical results and provide sharp upper and lower bounds for the payoffs in the Harsanyi set. In Section 4 we characterize the Weber set as subset of the Harsanyi set. The characterization proceeds by a number of steps, given in some lemma's and a proposition. The proofs of these steps are given in the Appendix. In Section 5 we prove the axiomatization of the Harsanyi set. Finally in Section 6 we consider the Harsanyi Imputation set.

2 Basic concepts

A cooperative game with transferable utilities, or simply a TU-game, is a pair (N, v), where $N = \{1, \ldots, n\}$ is a finite set of players and $v \colon 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$ is the characteristic function yielding for any subset S of N the payoff $v(S)$ that can be achieved by coalition S. Throughout the paper we use the notation $|S|$ for the number of players in coalition $S \subseteq N$. Unless otherwise stated, the set $N = \{1, \ldots, n\}$ is taken to be a fixed set of n players and we denote the set V to be the set of all games with player set N, i.e. V is the collection of all characteristic functions v on 2^N. Moreover, we denote $\Omega = 2^N$ as the collection of all subsets of N and define $\Omega_k = \{S \in \Omega| \ |S| = k\}$ as the collection of all subsets of N of size k, $0 \leq k \leq n$. Further, for all $S \in \Omega$ we define the set $\Omega^S = \{T \in \Omega | S \subseteq T\}$ as the collection of all subsets of N containing coalition S. In particular, $\Omega^i \equiv \Omega^{\{i\}} = \{T \in \Omega | i \in T\}$ is the collection of all subsets of N containing player i, $i \in N$. A payoff vector is a vector $x \in \mathbb{R}^n$ assigning payoff $x_i \in \mathbb{R}$ to player $i \in N$. For a payoff vector $x \in \mathbb{R}^n$, we denote by $x(S) = \sum_{i \in S} x_i$ as the total payoff to the players in coalition $S \in \Omega$.

A game $v \in V$ is called *convex* if for every pair $S, T \in \Omega$ holds $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$. A game $v \in V$ is *superadditive* if for every pair $S, T \in \Omega$ such that $S \cap T = \emptyset$ holds $v(S \cup T) \geq v(S) + v(T)$. In particular it follows that a convex game is superadditive and that for all $S \in \Omega$, $v(S) \geq \sum_{i \in S} v(i)$ when v is superadditive, where $v(i) = v(\{i\})$ is the payoff that player i can guarantee herself without cooperating with the other players. For $T \in \Omega$, the game u^T defined by $u^T(S) = 1$ if $T \subseteq S$ and $u^T(S) = 0$ otherwise is the *unanimity game* with respect to coalition T. Every unanimity game is convex. A game $v \in V$ is the *null game* if $v = v^0$ given by $v^0(S) = 0$ for all $S \in \Omega$.

Throughout this paper we assume that the grand coalition forms and therefore restrict ourselves to solution concepts in which the worth $v(N)$ of the grand coalition is distributed amongst its members.[1] A solution vector is said to be individually rational if any player gets at least her own worth $v(i)$. For a game $v \in V$, the set $I(v)$ of imputations is the set of all individually rational payoff vectors that efficiently distribute the payoff $v(N)$ of the grand coalition amongst its members, i.e.

$$I(v) = \{x \in \mathbb{R}^n | x(N) = v(N), \ x_i \geq v(i), \ i \in N\}.$$

For a game $v \in V$, $S \in \Omega$ and $i \in S$, the marginal value $z_i^S(v)$ is the contribution of player i to coalition S in game v when she is the last player joining S and thus is given by

$$z_i^S(v) = v(S) - v(S \setminus \{i\}).$$

For a permutation $\pi \colon N \to N$, assigning rank number $\pi(i) \in N$ to any player $i \in N$, define $S_i^\pi = \{j \in N | \pi(j) \leq \pi(i)\}$, so S_i^π is the set of all players with rank number at most equal to the rank number of i, including i herself and thus $S_i^\pi \in \Omega^i$. Then the *marginal value vector* $m^\pi(v) \in \mathbb{R}^n$ of game v and permutation π is given by

$$m_i^\pi(v) = z_i^{S_i^\pi}(v), \quad i \in N,$$

[1]Since we don't assume superadditivity, this does not guarantee efficiency of a solution, because there may exist partitions $\{S_1, \ldots, S_k\}$ of N such that $\sum_{j=1}^k v(S_j) > v(N)$.

and thus assigns to player i its marginal contribution to the worth of the coalition consisting of all its predecessors in π. The *Shapley value*, introduced by Shapley [10], is the solution function which assigns to each $v \in V$ the payoff vector $\psi^{Sh}(v)$ being the average of the marginal value vectors over all permutations, i.e.

$$\psi_i^{Sh}(v) = \frac{1}{n!} \sum_{\pi} m_i^{\pi}(v), \quad i \in N.$$

The *Weber mapping*, introduced by Weber [18], is the solution mapping which assings to each $v \in V$ the Weber set $W(v)$ of payoff vectors, being the convex hull of all marginal value vectors, i.e.

$$W(v) = \mathrm{Conv}\{m^{\pi}(v) | \pi \in \Pi\},$$

where Π is the set of all permutations on N. So, any convex combination of the marginal value vectors is in the Weber set and in particular the Shapley value $\psi^{Sh}(v) \in W(v)$. Clearly, for any $x \in W(v)$ holds $\sum_{i \in N} x_i = v(N)$, thus x is feasible and efficiently distributes the worth of the grand coalition N amongst its members. The Core, introduced in game theory by Gillies [2], is the solution mapping $C: V \to \mathbb{R}^n$ defined by

$$C(v) = \{x \in I(v) | x(S) \geq v(S), \ S \in \Omega\}.$$

So the Core is the set of imputations that cannot be improved upon by any coalition $S \subseteq N$ by distributing only their own value $v(S)$. It is well-known that the Shapley value is the barycenter of the Core when v is convex, see Shapley [11] and Ichiishi [6]. Furthermore $C(v) = W(v)$ when v is convex.

The dividends $\Delta^S(v)$, $S \in \Omega$, of the game v, as defined by Harsanyi [4], [5], follow recursively from the system of equations

$$v(S) = \sum_{T \subseteq S} \Delta^T(v), \quad S \in \Omega.$$

It is well-known that

$$v = \sum_{S \in \Omega} \Delta^S(v) u^S,$$

with u^S the *unanimity game* with respect to coalition S. So, any game v can be written as a linear combination of the characteristic functions of the unanimity games. Observe that the characteristic functions of the unanimity games are linearly independent, i.e. the characteristic functions of the unanimity games form a basis for the vector space V of characteristic functions (see e.g. Rosenmüller [8], Owen [7]). Hence, the dividends $\Delta^S(v)$, $S \in \Omega$, of the game v can be found alternatively as the uniquely determined coefficients v_S in the representation $v = \sum_{S \in \Omega} v_S u^S$. A game $v \in V$ is said to be *totally positive* (see

Vasil'ev [12]) if $\Delta^S(v) \geq 0$ for all $S \in \Omega$. The set of totally positive games is denoted by V^+ and is the closed convex cone in the vector space V. Further, we denote $V^- = -V^+$ as the set of totally negative games. Two games v and u are *disjoint* (see Vasil'ev [12]) if for any $S \in \Omega$ holds $\Delta^S(v) \cdot \Delta^S(u) = 0$. Given a game $v \in V$, we define the two totally positive games v^+ and v^- by

$$v^+ = \sum_{S|\Delta^S(v)>0} \Delta^S(v)u^S \text{ and } v^- = \sum_{S|\Delta^S(v)<0} -\Delta^S(v)u^S,$$

where the sum over the empty set is defined to be equal to the zero game v^0. Clearly, $v = v^+ - v^-$ and the pair (N, v^+) and (N, v^-) consists of disjoint games. From the convexity of the unanimity games it follows that any totally positive game, being a nonnegative linear combination of convex unanimity games, is also convex.

3 The Harsanyi set

In this section we consider the set of Harsanyi payoff vectors defined as the set of all payoof vectors obtained by distributing the dividend of a set S over the players in S for any $S \in \Omega$. Therefore, let P be the set of *dividend* share systems p_i^S, $S \in \Omega$, $i \in S$, given by

$$P = \{p = [p_i^S]_{i\in S}^{S\in\Omega} | p_i^S \geq 0,\ S \in \Omega \text{ and } i \in S,\ \sum_{i\in S} p_i^S = 1,\ S \in \Omega\}.$$

For $p \in P$, let the payoff vector $\phi^p(v) \in \mathbb{R}^n$ be given by

$$\phi_i^p(v) = \sum_{S\in\Omega^i} p_i^S \Delta^S(v),\ i \in N,$$

i.e. the payoff $\phi_i^p(v)$ to player $i \in N$ is the sum over all coalitions S containing i of the dividend share p_i^S of player i in the Harsanyi dividend $\Delta^S(v)$ of coalition S. We therefore call $\phi_i^p(v)$ a *Harsanyi payoff vector*. Observe that, due to the equality $v(N) = \sum_{S\in\Omega} \Delta^S(v)$, for any weight system $p \in P$ holds

$$\sum_{i\in N} \phi_i^p(v) = v(N)$$

and thus any Harsanyi payoff vector is feasible, i.e. the Harsanyi payoffs distribute the total value $v(N)$ over the players in N. In particular it is well-known that $\phi^p(v)$ is equal to the Shapley value $\psi^{Sh}(v)$ when

$$p_i^S = \frac{1}{|S|},\ \text{for } S \in \Omega \text{ and } i \in S.$$

The *Harsanyi mapping* is the solution mapping which assings to each $v \in V$ the *Harsanyi set* $H(v)$ of payoff vectors given by

$$H(v) = \{\phi^p(v) \in \mathbb{R}^n | p \in P\}.$$

So, the Harsanyi set is the set of all distributions of the total payoff $v(N)$ over the players in N that can be generated by some weight system $p \in P$. The Harsanyi set as defined above is the same as the selectope discussed recently by Derks, Haller and Peters [1] and introduced much earlier already by Hammer, Peled and Sorensen [3]. The notion of Harsanyi set and related concepts have independently been introduced by Vasil'ev [12] and [13] (in Russian) and also in Vasil'ev [14]. In this paper we prefer the use of Harsanyi set instead of selectope, because we want to stress the property of distributing the Harsanyi dividends instead of the role of the selectors as discussed by Derks et al [1].

In the remaining of this section we consider the structure of the Harsanyi set. First, for historical reasons we recall some results (Lemma 3.1, Theorem 3.2, Corollary 3.3 and Corollary 3.4), which have been given already in Vasil'ev [14], [15] (in Russian) and Derks et al [1] and also in the earlier papers of Vasil'ev [13] (in Russian) and Hammer et al [3]. We refer to these papers for the proofs. In particular these results show that the Harsanyi set has a Core-type structure. First, for a game $v \in V$, the corresponding game $v_H \in V$ is defined by

$$v_H(S) = v(S) - [v^-(N) - v^-(S) - v^-(N \setminus S)],\ S \in \Omega.$$

As will become clear later, we call this game v_H the *Harsanyi mingame*. The next lemma is useful to prove the succeeding theorem.

Lemma 3.1 *For any $v \in V$, the Harsanyi mingame v_H is convex.*

Theorem 3.2 *For any $v \in V$ holds $H(v) = C(v_H)$.*

The theorem shows that the Harsanyi set can be found easily as the Core of the Harsanyi mingame v_H. Moreover, since v_H is convex, it follows that the Harsanyi set is equal to the Weber set of v_H and thus equal to the convex hull of the marginal value vectors of v_H. Since any totally positive game is convex it also follows that for any $S \in \Omega$ holds $v^-(N) \geq v^-(S) + v^-(N \setminus S)$ and thus $v_H(S) \leq v(S)$. Clearly, we also have $v_H(N) = v(N)$. This implies the next corollary, which says that the Core of a game is a subset of the Harsanyi set.

Corollary 3.3 *For any $v \in V$ holds $C(v) \subseteq H(v)$.*

For a game $v \in V$, let the *normalised v-game* $\widehat{v} \in V$ be defined by

$$\widehat{v}(S) = v(S) - \sum_{i \in S} v(i),\ S \in \Omega,$$

i.e. $\widehat{v}$ is obtained from v by normalising all worths $v(i)$, $i \in S$, to zero. Then the next corollary says that the Harsanyi set of a game v is equal to the Core of v if and only if $\widehat{v}$

is totally positive and follows immediately from Theorem 3.2 by observing that $v_H = v$ if $v \in V^+$ and that both $C(v) = \{v^1\} + C(\widehat{v})$ and $H(v) = \{v^1\} + H(\widehat{v})$, where $v^1 \in \mathbb{R}^n$ is the n-dimensional vector given by $v^1_i = v(i),\ i \in N$.

Corollary 3.4 *$H(v) = C(v)$ if and only if $\widehat{v} \in V^+$.*

The next theorem relates the minimum and maximum payoff that a coalition can obtain at a payoff vector in the Harsanyi set to the worths of the coalition in the Harsanyi mingame v_H as defined above and the similarly defined *Harsanyi maxgame* $v^H \in V$ given by

$$v^H(S) = v(S) + [v^+(N) - v^+(S) - v^+(N \setminus S)],\ S \in \Omega.$$

Theorem 3.5 *For any $v \in V$ holds that*

$$v_H(S) = \min\{x(S) | x \in H(v)\}, \quad S \in \Omega,$$

and

$$v^H(S) = \max\{x(S) | x \in H(v)\}, \quad S \in \Omega.$$

Proof.
For $S \in \Omega$, denote $\Omega(S) = \{T \in \Omega | T \cap S \neq \emptyset \text{ and } T \setminus S \neq \emptyset\}$ and $\Omega^i(S) = \Omega^i \cap \Omega(S)$. Applying the equality $v(S) = \sum_{T \subseteq S} \Delta^T(v)$ we have for $p \in P$ and $S \subseteq N$ that

$$\begin{aligned} \sum_{i \in S} \phi^p_i(v) &= \sum_{i \in S} \sum_{T \in \Omega^i} p^T_i \Delta^T(v) = \sum_{T \subseteq S} \sum_{i \in T} p^T_i \Delta^S(v) + \sum_{i \in S} \sum_{T \in \Omega^i(S)} p^T_i \Delta^T(v) \\ &= v(S) + \sum_{i \in S} \sum_{T \in \Omega^i(S)} p^T_i \Delta^T(v). \end{aligned}$$

It follows straightforward that the last term in this expression is bounded from below by

$$\sum_{i \in S} \sum_{T \in \Omega^i(S)} p^T_i \Delta^T(v) \geq - \sum_{T \in \Omega(S)} \Delta^T(v^-)$$

with equality for $\underline{p} \in P$ satisfying for each $T \in \Omega(S)$

$$\underline{p}^T_i = 1 \text{ if } i = i_T \text{ and } \underline{p}^T_i = 0 \text{ otherwise,}$$

where

$$\begin{aligned} i_T &= \max\{j \mid j \in T \setminus S\} \text{ if } \Delta^T(v) \geq 0, \\ &= \min\{j \mid j \in T \cap S\} \text{ if } \Delta^T(v) < 0. \end{aligned}$$

Also the last term is bounded from above by

$$\sum_{i \in S} \sum_{T \in \Omega^i(S)} p^T_i \Delta^T(v) \leq \sum_{T \in \Omega(S)} \Delta^T(v^+)$$

with equality for $\overline{p} \in P$ satisfying for each $T \in \Omega(S)$

$$\overline{p}_i^T = 1 \text{ if } i = j_T \text{ and } \overline{p}_i^T = 0 \text{ otherwise,}$$

where

$$\begin{aligned} j_T &= \max\{j \mid j \in T \cap S\} \text{ if } \Delta^T(v) > 0, \\ &= \min\{j \mid j \in T \setminus S\} \text{ if } \Delta^T(v) \leq 0. \end{aligned}$$

Hence, for any $S \in \Omega$ we have

$$\begin{aligned} \min\{\sum_{i \in S} \phi_i^p(v) | p \in P\} &= v(S) - \sum_{T \in \Omega(S)} \Delta^T(v^-), \\ \max\{\sum_{i \in S} \phi_i^p(v) | p \in P\} &= v(S) + \sum_{T \in \Omega(S)} \Delta^T(v^+). \end{aligned}$$

Therefore, together with

$$\sum_{T \in \Omega(S)} \Delta^T(v^-) = v^-(N) - v^-(S) - v^-(N \setminus S) = v(S) - v_H(S)$$

and

$$\sum_{T \in \Omega(S)} \Delta^T(v^+) = v^+(N) - v^+(S) - v^+(N \setminus S) = -v(S) + v^H(S)$$

it follows that for any $S \in \Omega$ holds

$$v_H(S) = \min\{x(S) | x \in H(v)\} \text{ and } v^H(S) = \max\{x(S) | x \in H(v)\}.$$

Q.E.D.

Theorem 3.5 shows that the minimum, respectively maximum payoffs that the coalitions $S \in \Omega$ can obtain at a payoff vector in the Harsanyi set are given by the characteristic functions of the Harsanyi mingame and the Harsanyi maxgame, so that these games provide sharp lower and upper bounds for the total payoff of a coalition in the Harsanyi set. Besides the calculation of these lower and upper bounds given in the proof above these bounds also follow directly from the fact the v_H is convex and therefore exact, i.e. for any $S \in \Omega$, there exists $x \in C(v_H)$, such that $x(S) = v_H(S)$, see Schmeidler [9]. Since $H(v) = C(v_H)$ it follows that

$$v_H(S) = \min\{x(S) | x \in C(v_H)\} = \min\{x(S) | x \in H(v)\}, \ S \in \Omega,$$

which shows that the lower bound is sharp. To show the sharp upper bound, it follows from the definition of v_H and v^H that

$$v_H(N \setminus S) + v^H(S) = v(N), \ S \in \Omega.$$

Take $S \in \Omega$ and $\widehat{x} \in H(v)$, such that $\widehat{x}(N \setminus S) = v_H(N \setminus S)$. Then

$$\widehat{x}(S) = v(N) - \widehat{x}(N \setminus S) = v(N) - v_H(N \setminus S) = v^H(S).$$

Moreover, for any $x \in H(V) = C(v_H)$, we have that $x(N \setminus S) \geq v_H(N \setminus S)$ and thus $x(S) = v(N) - x(N \setminus S) \leq v(N) - v_H(N \setminus S) = v^H(S)$, so that the upper bound holds for any x with equality for $\widehat{x}$.

4 Characterization of the Weber set by Harsanyi payoff vectors

In this section we consider the relation between the Harsanyi set and the Weber set. Derks et al [1] have shown that the Weber set is a subset of the Harsanyi set and also provide conditions on the game v under which the Harsanyi set of v coincides with the Weber set. In the next theorem we give a characterization for any $v \in V$ the Weber set as a subset of the Harsanyi set consisting of all Harsanyi payoff vectors corresponding to the dividend share systems in the subset P^W of P given by

$$P^W = \{p \in P | \sum_{T \in \Omega^S} (-1)^{|T|-|S|} p_i^T \geq 0,\ S \in \Omega^i,\ i \in N\}.$$

Theorem 4.1 *For any game $v \in V$, the Weber set $W(v)$ is the subset of the Harsanyi set $H(v)$ given by*

$$W(v) = \{\phi^p(v) \in H(v) |\ p \in P^W\}.$$

To prove the theorem, we have to introduce some definitions. First, let Q be the collection of all systems $q_i^S,\ S \in \Omega,\ i \in S$, assigning a 'weight' $q_i^S \in \mathbb{R}$ to any player i in any coalition $S \in \Omega$, i.e.

$$Q = \{q | q = [q_i^S]_{i \in S}^{S \in \Omega}\}.$$

Observe that the set of dividend share systems P is a subset of the set of all weight systems Q. For $q \in Q$ and $S \in \Omega$, to shorten notation we denote the sum of the weights of the players in S by $q(S)$, i.e.

$$q(S) = \sum_{i \in S} q_i^S.$$

Next, we define the two subsets Q_1 and Q_1^* of Q by

$$Q_1 = \{q \in Q | \sum_{i \in S} \sum_{T \in \Omega^S} q_i^T = 1,\ S \in \Omega\},$$

and

$$Q_1^* = \{q \in Q | q(N) = 1 \text{ and } q(S) = \sum_{j \in N \setminus S} q_j^{S \cup j},\ S \in \Omega,\ S \neq N\},$$

where $S \cup j$ denotes $S \cup \{j\}$. Next, we define $Q^W \subseteq Q_1$ by

$$Q^W = \{q \in Q_1 | q_i^S \geq 0,\ i \in S,\ S \in \Omega\}.$$

Clearly, both P^W and Q^W are two polyhedra in the vector space Q of all weight systems. For a polyhedron $R \subseteq Q$, let $\mathrm{Ex}(R)$ be the set of all extreme points of R. Further, for any permutation $\pi = (\pi(1), \ldots, \pi(n)) \in \Pi$ and $S \in \Omega$, recall that $S_i^\pi = \{j \in N | \pi(j) \leq \pi(i)\}$. Now, for any $\pi \in \Pi$, we define the weights systems $p^\pi \in Q$ and $q^\pi \in Q$ by setting for all $S \in \Omega$ and $i \in S$,

$$\begin{aligned} p_i^{\pi,S} &= 1,\ \text{if } S \subseteq S_i^\pi \\ &= 0,\ \text{otherwise;} \end{aligned}$$

and

$$\begin{aligned} q_i^{\pi,S} &= 1,\ \text{if } S = S_i^\pi, \\ &= 0,\ \text{otherwise.} \end{aligned}$$

Finally, let the two linear functions $F\colon Q \to Q$ and $G\colon Q \to Q$ be defined by

$$F_i^S(q) = \sum_{T \in \Omega^S} (-1)^{|T|-|S|} q_i^T,\ i \in N,\ S \in \Omega^i,$$

and

$$G_i^S(q) = \sum_{T \in \Omega^S} q_i^T,\ i \in N,\ S \in \Omega^i.$$

We now proceed by a number of steps, given in the following lemma's and the main result given in Proposition 4.5. For the proofs we refer to the Appendix. Recall that $\Omega_k = \{S \in \Omega |\ |S| = k\}$.

Lemma 4.2 *For any $n \geq 1$ holds that $Q_1 = Q_1^*$.*

Lemma 4.3 *For any $q \in Q^W$ holds*

$$\sum_{S \in \Omega_k} q(S) = 1,\ k = 1, \ldots, n.$$

Lemma 4.4 *For all $\pi \in \Pi$ holds:*

(i) $\quad q^\pi \in Ex(Q^W)$,

(ii) $p^\pi \in Ex(P^W)$,

(iii) $F(p^\pi) = q^\pi$,

(iv) $G(q^\pi) = p^\pi$.

Using the results of Lemma 4.2 and Lemma 4.4, the next proposition has been proven in Vasil'ev [16]. This proposition plays a central role in proving the main result given in Theorem 4.1 and is interesting in itself because it characterizes the extreme points of the set Q^W. In the Appendix the original proof given in Vasil'ev [16] is completely renewed and substantially shortened by using the property of Q^W given in the new Lemma 4.3.

Proposition 4.5 *For any $n \geq 1$ holds $Ex(Q^W) = \{q^\pi | \pi \in \Pi\}$,*

Finally we prove in the Appendix the next lemma's.

Lemma 4.6 *The functions F and G satisfy*

(i) *G is a nondegenerate linear operator,*

(ii) $F = G^{-1}$,

(iii) $G(Q^W) = P^W$.

Lemma 4.7 *For all $v \in V$ and any $\pi \in \Pi$ holds*

$$\phi^{p^\pi}(v) = m^\pi(v).$$

The proof of Theorem 4.1 now follows easily from Proposition 4.5, Lemma 4.6 and Lemma 4.7.

Proof of Theorem 4.1.
Since for any nondegenerate linear operator Φ it is well-known that the image $B = \Phi(A)$ of any convex polyhedron A is again a convex polyhedron with the extreme points coinciding with the images of the extreme points of A, i.e. $\mathrm{Ex}(B) = \{\Phi(a) | a \in \mathrm{Ex}(A)\}$, it follows immediately from Proposition 4.5 and Lemma 4.6 that $\mathrm{Ex}(P^W) = \{p^\pi | \pi \in \Pi\}$. So, for $p \in P^W$, there exist $\lambda_\pi \geq 0$ with $\sum_{\pi \in \Pi} \lambda_\pi = 1$ such that

$$p = \sum_{\pi \in \Pi} \lambda_\pi p^\pi.$$

To simplify notation, denote $\phi^p(v) = f(p, v)$. Then, together with Lemma 4.7 it follows that

$$\phi^p(v) = f(p, v) = f(\sum_{\pi \in \Pi} \lambda_\pi p^\pi, v) = \sum_{\pi \in \Pi} \lambda_\pi f(p^\pi, v) = \sum_{\pi \in \Pi} \lambda_\pi m^\pi(v)$$

and thus $f(p,v) = \phi^p(v) \in W(v)$. On the other hand, suppose $x \in W(v)$. Then there exist $\lambda_\pi \geq 0$ with $\sum_{\pi \in \Pi} \lambda_\pi = 1$ such that

$$x = \sum_{\pi \in \Pi} \lambda_\pi m^\pi(v) = \sum_{\pi \in \Pi} \lambda_\pi f(p^\pi, v) = f(\sum_{\pi \in \Pi} \lambda_\pi p^\pi, v) = f(p,v) = \phi^p(v)$$

with $p = \sum_{\pi \in \Pi} \lambda_\pi p^\pi \in P^W$. Q.E.D.

To conclude the characterization of the Weber set, we define for $q \in Q^W$ the payoff vector $\psi^q(v) \in \mathbb{R}^n$ by

$$\psi_i^q(v) = \sum_{S \in \Omega^i} q_i^S z_i^S(v), \;\; i \in N,$$

i.e. the payoff $\psi_i^q(v)$ to player $i \in N$ is the sum over all coalitions S containing i of the weight q_i^S of player i in the marginal value $z_i^S(v)$ of i to coalition S. Recall that any $q \in Q^W$ satisfies the condition

$$\sum_{i \in S} \sum_{T \in \Omega^S} q_i^T = 1$$

for all $S \in \Omega$. Taking $S = \{i\}$ we obtain

$$\sum_{T \in \Omega^i} q_i^T = 1,$$

i.e. for any $i \in N$ we have that the sum of the weights of i in the coalitions containing i is equal to one, implying that $\psi_i^q(v)$ is a weighted sum of the marginal contributions of player i. It follows straightforward, see for instance Vasil'ev [15] that for any $q \in Q^W$ it holds that

$$\psi^q(v) = \phi^p(v)$$

with $p = G(q) \in P^W$. Since $\sum_{i \in N} \phi_i^p(v) = v(N)$ for any $p \in P^W$ it follows that

$$\sum_{i \in N} \psi_i^q(v) = \sum_{i \in N} \phi_i^{G(q)}(v) = v(N)$$

and thus for any $q \in Q^W$ the payoff vector $\psi_i^q(v)$ distributes the total payoff $v(N)$ over the players in N. In particular it is well-known that $\psi^{\overline{q}}(v) = \psi^{Sh}(v)$ for the weight system $\overline{q} \in Q^W$ given by

$$\overline{q}_i^S = \frac{(|S|-1)!(n-|S|)!}{n!}, \;\; i \in S, \;\; S \in \Omega.$$

5 Axiomatization of the Harsanyi set

In this section we provide an axiomatization of the Harsanyi solution mapping set $H\colon V \to \mathbb{R}^n$, i.e. the value mapping assigning the Harsanyi set $H(v)$ to any $v \in V$. To do so, first we state six reasonable axioms to be satisfied by a value mapping $M\colon V \to \mathbb{R}^n$, see also Vasil'ev [14].

Axiom M

(i) A value mapping M is *convex* if $M(v)$ is convex for any $v \in V$.

(ii) A value mapping M has the *efficient distribution property* when for any $x \in M(v)$ holds $\sum_{i \in N} x_i = v(N)$.[2]

(iii) A value mapping M has the *dummy player property* when for any $i \in N$ such that $v(S) = v(S \setminus \{i\})$ for all $S \in \Omega^i$, holds $x_i = 0$ for all $x \in M(v)$.

(iv) A value mapping M is *sign preserving* if $M(v) \subseteq \mathbb{R}^n_+$ for any totally positive game $v \in V^+$ and $M(v) \subseteq \mathbb{R}^n_-$ for any totally negative game $v \in V^-$.

(v) A value mapping M is *individually consistent* if for any $v \in V$ and any $i \in N$ holds $M(v_i) \subseteq M(v)$, where $v_i \in V$ is defined by $v_i(S) = v(S \setminus \{i\}) + v(N) - v(N \setminus \{i\})$ if $S \in \Omega^i$ and $v_i(S) = v(S)$ otherwise.

(vi) A value mapping M is *disjoint additive* if for any disjoint pair v and u of games holds $M(v + u) = M(v) + M(u)$.

We now have the following theorem.

Theorem 5.1
A mapping $M\colon G^N \to \mathbb{R}$ satisfies (i)-(vi) of Axiom M if and only if $M(v) = H(v)$ for all $v \in V$.

Proof. We first prove that the Harsanyi mapping satisfies all axioms (i)-(vi). Recall that for $v \in V$, $H(v) = \{\phi^p(v) \in \mathbb{R}^n | p \in P\}$, where $P = \{p = [p_i^S]_{i \in S}^{S \in \Omega} | p_i^S \geq 0,\ S \in \Omega$ and $i \in S,\ \sum_{i \in S} p_i^S = 1,\ S \in \Omega\}$ and $\phi_i^p(v) = \sum_{S \in \Omega^i} p_i^S \Delta^S(v),\ i \in N$. Now, for $S \in \Omega$, let $A(S) \subset \mathbb{R}^n$ be given by

$$A(S) = \{p \in \mathbb{R}^n_+ | p_j = 0,\ j \in N \setminus S \text{ and } \sum_{i \in S} p_i = 1\},$$

[2] This axiom requires that $v(N)$ is distributed efficiently amongst its members, not that $M(v)$ is a subset of the set of efficient payoff vectors. Recall that it is assumed that the grand coalition forms and $v(N)$ has to be distributed.

i.e. $A(S)$ is the set of all feasible dividend share vectors within the coalition S extended to an n-dimensional vector by setting all components with respect to $j \notin S$ equal to zero. Clearly $\Delta^S(v)A(S)$ is the set of all possible distributions of the dividend $\Delta^S(v)$ over the players in S. Consequently, it follows that $H(v)$ can be written as

$$H(v) = \sum_{S \in \Omega} \Delta^S(v)A(S). \tag{1}$$

From this representation of $H(v)$ axiom (i) follows immediately from the convexity of each set $A(S)$.

To show that H satisfies axiom (ii), observe that for any $x \in H(v)$, it follows from (1) that there exist $p^{S,x} \in A(S)$, $S \in \Omega$ such that

$$x = \sum_{S \in \Omega} \Delta^S(v)p^{S,x}$$

Since by definition of $A(S)$ we have that $p^{S,x}(N) = \sum_{i \in N} p_i^{S,x} = 1$ for all $S \in \Omega$, we get that

$$x(N) = \sum_{i \in N} x_i = \sum_{S \in \Omega} \Delta^S(v)p^{S,x}(N) = \sum_{S \in \Omega} \Delta^S(v) = v(N).$$

To show axiom (iii), suppose that i is a dummy player, so $v(S) = v(S \setminus \{i\})$ for all $S \in \Omega^i$. It is well-known fact that then the dividend $\Delta^S(v) = 0$ for all $S \in \Omega^i$. Since for any $p \in A(S)$ we have that $p_i = 0$ if $S \in \Omega \setminus \Omega^i$ it follows from (1) that $x_i = 0$ for all $x \in H(v)$.

Axiom (iv) follows immediately from (1) because $\Delta^S(v) \geq 0$ for all $S \in \Omega$ when $v \in V^+$ and $\Delta^S(v) \leq 0$ for all $S \in \Omega$ when $v \in V^-$.

To prove that axiom (v) is satisfied, it follows straightforward from its definition that the dividends of the game v_i are given by

$$\begin{aligned}
\Delta^S(v_i) &= \sum_{T \in \Omega^i} \Delta^T(v), \text{ if } S = \{i\}, \\
&= \Delta^S(v), \text{ if } S \subseteq N \setminus \{i\}, \\
&= 0, \text{ otherwise}
\end{aligned}$$

For any $p \in P$, let $\phi^p(v_i)$ be the corresponding payoff vector in $H(v_i)$. It follows from the dividends given above that $\phi^p(v_i) = \phi^{\overline{p}}(v) \in H(v)$ where

$$\begin{aligned}
\overline{p}_j^S &= p_j^S, \ j \in S, \ S \subseteq N \setminus \{i\}, \\
&= 1, \ j = i, \ S \in \Omega^i, \\
&= 0, \ j \neq i, \ j \in S \in \Omega^i.
\end{aligned}$$

Hence $H(v_i) \subseteq H(v)$.

Finally, to show axiom (vi), let v and w be a disjoint pair of two games in V and define $z = v + w$. Then it is well-known that $\Delta^S(z) = \Delta^S(v) + \Delta^S(w)$ for any $S \in \Omega$. Since v and w are disjoint and thus $\Delta^S(v) \cdot \Delta^S(w) = 0$ for any $S \in \Omega$, it follows from applying (1) that[3]

$$\begin{aligned} H(z) &= \sum_{S\in\Omega} \Delta^S(z)A(S) \\ &= \sum_{S\in\Omega|\Delta^S(v)\neq 0} \Delta^S(v)A(S) + \sum_{S\in\Omega|\Delta^S(w)\neq 0} \Delta^S(w)A(S) = H(v) + H(w). \end{aligned}$$

Thus the mapping H is disjoint additive.
It remains to show that H is the only mapping that satisfies the axioms (i)-(vi). To do so, recall that for any game $v \in V$ holds

$$v = \sum_{S\in\Omega} \Delta^S(v)u^S.$$

Let M be a mapping satisfying the axioms (i)-(vi). For the unanimity game u^S, $S \in \Omega$, let $x \in M(u^S)$. Then axiom (ii) requires that $\sum_{i\in N} x_i = u^S(N) = 1$, axiom (iii) that $x_j = 0$ for any $j \in N \setminus S$ and axiom (iv) that $x \in \mathbb{R}^n_+$. Moreover, according to axiom (v) we have that $e(i) \in M(u^S)$, $i \in S$, where $e(i)$ is the i-th unit vector, i.e. the vector with i-th component equal to one and all other components equal to zero. Together with the convexity axiom (i) it follows that for the unanimity game u^S, $M(u^S) = A(S)$ is the unique set of payoff vectors satisfying the axioms (i)-(v). Since any pair (u^S, u^T), $S \neq T$ is disjoint, it follows from axiom (vi) that the mapping $M: V \to \mathbb{R}^n$ satisfying (i)-(vi) is uniquely determined by

$$M(v) = \sum_{S\in\Omega} \Delta^S(v)A(S) = H(v).$$

Q.E.D.

6 The Harsanyi Imputation set

In this section we consider the Harsanyi Imputation set $H^I(v)$ of a game $v \in V$, being the subset of the Harsanyi set of all individually rational payoff vectors, i.e.

$$H^I(v) = H(v) \cap I(v)$$

First of all, since $C(v) \subseteq I(v)$ it follows from $C(v) \subseteq H(v)$ that also $C(v) \subseteq H^I(v) = H(v) \cap I(v)$, i.e. any payoff vector in the Core is a Harsanyi imputation. So, the Core of v

[3]Observe that it is not allowed to split up $\sum_{S\in\Omega} \Delta^S(z)A(S)$ when there are some S such that $\Delta^S(v) \cdot \Delta^S(w) \neq 0$, i.e. when v and w are not disjoint.

serves as a *lower bound* on the Harsanyi Imputation set in the sense that $C(v)$ is contained in $H^I(v)$. For a further characterization of the the Harsanyi Imputation set we define for $v \in V$ the corresponding *Harsanyi v-game* $\overline{v}_H \in V$ by

$$\overline{v}_H(S) = \sum_{i \in S} v(i) + \max_{T \subseteq S} \widehat{v}_H(T), \quad S \in \Omega,$$

where $\widehat{v}_H$ is the Harsanyi mingame corresponding to the normalized v-game $\widehat{v}$. So, in the Harsanyi v-game the worth of a coalition is the total worth of its members plus the maximum over the worths of its subcoalitions in the corresponding Harsanyi mingame of the normalised v-game. The next lemma and theorem are given by Vasil'ev [14].

Lemma 6.1 *For any $v \in V$, the Harsanyi v-game $\overline{v}_H$ is convex.*

Theorem 6.2 *For any $v \in V$ holds*

$$\begin{aligned} H^I(v) &= C(\overline{v}_H), \text{ if } v^H(S) \geq \sum_{i \in S} v(i) \text{ for all } S \in \Omega, \\ &= \emptyset, \text{ otherwise.} \end{aligned}$$

The theorem says that the Harsanyi set is either empty or equal to the Core of the Harsanyi v-game $\overline{v}_H$. Moreover, since $\overline{v}_H$ is convex, it follows that in the latter case the Harsanyi set is equal to the Weber set $W(\overline{v}_H)$ of $\overline{v}_H$ and thus equal to the convex hull of the marginal value vectors of $\overline{v}_H$. Clearly, the theorem implies that $v(N) = \overline{v}_H(N)$ when $H^I(v)$ is not empty. In fact, it follows immediately from the definition of $\overline{v}_H$ that for any v it holds that $\overline{v}_H(N) \geq v(N)$. It can be shown easily that this holds with equality if and only if $v^H(S) \geq \sum_{i \in S} v(i)$ for all $S \in \Omega$. This gives us the next corollary as an alternative formulation of Theorem 6.2.

Corollary 6.3 *For any $v \in V$ holds $v(N) \leq \overline{v}_H(N)$ and*

$$\begin{aligned} H^I(v) &= C(\overline{v}_H), \text{ if } v(N) = \overline{v}_H(N), \\ &= \emptyset, \text{ if } v(N) < \overline{v}_H(N). \end{aligned}$$

Since v^+ is totally positive and thus convex, it follows from the definition of v^H that $v^H(S) \geq v(S)$ for all $S \in \Omega$. So, the condition $v^H(S) \geq \sum_{i \in S} v(i)$ is satisfied when v is superadditive, which gives the next corollary.

Corollary 6.4 *For $v \in V$ holds that $H^I(v) \neq \emptyset$ when v is superadditive.*

From $C(v) \subseteq H^I(v)$ also the next corollary follows immediately.

Corollary 6.5 *For $v \in V$ holds that $H^I(v) = C(\overline{v}_H)$ when $C(v) \neq \emptyset$.*

By theorem 4.1 we know that $W(v) \subseteq H(v)$. So, when $H^I(v)$ is empty, any payoff vector in the Weber set and thus also the Shapley value does not belong to the Imputation set. Together with Theorem 6.2 this gives the next corollary.

Corollary 6.6 *For $v \in V$ holds $W(v) \cap I(v) = \emptyset$ when there exists $S \in \Omega$ such that $v^H(S) < \sum_{i \in S} v(i)$.*

The next theorem states that $H^I(v)$ is *external stable* (in the sense of Von Neumann-Morgenstern [17]), when not empty. For an imputation $x \in I(v)$, we say that $y \in I(v)$ dominates x, denoted by $y \succ x$, when there exists a coalition S such that $y_i > x_i$ for all $i \in S$ and $\sum_{i \in S} y_i \leq v(S)$. A set $Y \subseteq I(v)$ is external stable when for every $x \in I(v) \setminus Y$ there exists $y \in Y$ such that y dominates x.

Theorem 6.7 *For any $v \in V$ holds that $H^I(v)$ is external stable, when $H^I(v)$ is not empty.*

Proof. Suppose $H^I(v) \neq \emptyset$. Since both $I(v) = I(\widehat{v}) + \{v^1\}$ and $H^I(v) = H^I(\widehat{v}) + \{v^1\}$, without loss of generality we may assume that v is normalized, i.e. $v(i) = 0$ for all $i \in N$, and thus $\widehat{v}(S) = v(S)$ for all $S \subset \Omega$. Then according to Theorem 6.2 we have that

$$v^H(S) \geq 0, \text{ for any } S \in \Omega.$$

By definition of v_H and v^H it follows that

$$v^H(N \setminus S) + v_H(S) = v(N), \;\; S \in \Omega$$

and thus

$$v_H(S) = v(N) - v^H(N \setminus S) \leq v(N), \;\; S \in \Omega.$$

By definition of v_H it holds that $v_H(N) = v(N)$ and thus

$$\overline{v}_H(N) = \max_{S \subseteq N} v_H(S) = v(N).$$

Observe also that for any $i \in N$, $v_H(i) = v(i) - v^-(N) + v^-(S) + v^-(N \setminus S) = -[v^-(N) - v^-(S) - v^-(N \setminus S)] \leq 0$, because $v(i) = 0$ and v^- is convex. Hence

$$\overline{v}_H(i) = \max_{S \subseteq \{i\}} v_H(i) = 0$$

because $v_H(\emptyset) = 0$. Therefore it follows from Theorem 6.2 that

$$H^I(v) = C(\overline{v}_H) = \{x \in \mathbb{R}^n_+ | \; x(N) = v(N) \text{ and } x(S) \geq \overline{v}_H(S) \text{ if } |S| \geq 2\}$$

and thus $H^I(v) \subset \mathbb{R}^n_+$.

When $I(v) \setminus H^I(v) = \emptyset$, the theorem is true. So, consider the case $I(v) \setminus H^I(v) \neq \emptyset$ and let y be any element belonging to $I(v) \setminus H^I(v)$. Since $y \notin H^I(v) = C(\overline{v}_H)$, there exists $S \in \Omega$, such that $y(S) < \overline{v}_H(S)$. Let S_0 be a minimal coalition satisfying this inequality, i.e. $y(S_0) < \overline{v}_H(S_0)$ and $y(T) \geq \overline{v}_H(T)$ for any $T \subset S_0$, $T \neq S_0$. Define $x^0 = (x_i^0)_{i \in S_0}$ by

$$x_i^0 = y_i + \delta, \; i \in S_0,$$

where $\delta = \frac{1}{|S_0|}(\overline{v}_H(S_0) - y(S_0))$. Clearly $x^0(S_0) = \sum_{i \in S_0} x_i^0 = \overline{v}_H(S_0)$ and thus $x^0 \in C(\overline{v}_H^0)$, where $\overline{v}_H^0$ is the restriction of $\overline{v}_H$ to S_0. By Lemma 6.1 the game $\overline{v}_H$ is convex and thus there exists an extension $x \in \mathbb{R}^n$ of x^0 in the Core of the game $\overline{v}_H$, i.e. a vector x such that $x_i = x_i^0$ for $i \in S_0$ and $x(N) = \overline{v}_H(N) = v(N)$, such that $x \in C(\overline{v}_H)$. Thus $x \in C(\overline{v}_H) = H^I(v)$ and satisfies

$$x_i = x_i^0 > y_i, \; i \in S_0, \text{ and } x(S_0) = \overline{v}_H(S_0).$$

To show that x dominates y through S_0 in the game v it remains to prove that $\overline{v}_H(S_0) \leq v(S_0)$. Since v^- is convex and thus superadditive, we have by definition of v_H that $v_H(S_0) \leq v(S_0)$. Now, suppose that $\overline{v}_H(S_0) > v_H(S_0)$. Then, by definition of $\overline{v}_H$, there exists $T \subset S_0$, $T \neq S_0$, such that $\overline{v}_H(S_0) = v_H(T)$. Since y is nonnegative, we have that

$$y(T) \leq y(S_0) < \overline{v}_H(S_0) = v_H(T) \leq \overline{v}_H(T),$$

which contradicts that $y(T) \geq \overline{v}_H(T)$. Hence $\overline{v}_H(S_0) = v_H(S_0) \leq v(S_0)$. Q.E.D.

Finally, let $C^*(v)$ be the classical Core of the game v being the set of undominated imputations, i.e.

$$C^*(v) = \{x \in I(v) | \text{ there is no } y \in I(v) \text{ such that } y \text{ dominates } x\}.$$

Then it follows from the external stability of the set $H^I(v)$ that the classical Core is contained in the Harsanyi Imputation set if the latter set is not empty. So, we have the following corollary.

Corollary 6.8 *For any $v \in V$ with $H^I(v) \neq \emptyset$ holds $C^*(v) \subseteq H^I(v)$.*

7 Appendix

To give the proofs, we first introduce some additional notation. First, for $q \in Q$ and $S \in \Omega$, recall that $q(S) = \sum_{i \in S} q_i^S$. Also, recall that $\Omega_k = \{S \in \Omega | \; |S| = k\}$. Further, for $S \in \Omega$ and $|S| \leq k \leq n$, we define

$$\Omega_k^S = \Omega^S \cap \Omega_k = \{T \in \Omega^S | \; |T| = k\},$$

i.e. Ω_k^S is the collection of all subsets of N of size k and containing S and

$$\Omega_{(k)}^S = \cup_{r=k}^n \Omega_r^S.$$

Moreover, we define

$$\Omega_S^k = \{T \subset N \setminus S \mid |T| = k\},$$

i.e. Ω_S^k is the collection of all subsets of $N \setminus S$ of size k.

Proof of Lemma 4.2.

For $n = 1$, the lemma is true by definition of Q_1 and Q_1^*. So, we suppose that $n \geq 2$. First we prove that $Q_1^* \subseteq Q$. Fix an arbitrary $q \in Q_1^*$ and $S \subseteq \Omega$. Let $m = |S|$ and define

$$A^S = \sum_{i \in S} \sum_{T \in \Omega^S} q_i^T$$

and for $k = m+1, m+2, \ldots, n$, define

$$A_k^S = \sum_{i \in S} \sum_{T \in \Omega_{(k)}^S} q_i^T$$

and $A_{n+1}^S \equiv 0$. We prove by induction that

$$A^S = \sum_{T \in \Omega_k^S} q(T) + A_{k+1}^S, \quad k = m, \ldots, n. \tag{2}$$

First, observe that Ω_m^S only contains the set S and thus

$$A^S = \sum_{i \in S} \sum_{T \in \Omega^S} q_i^T = q(S) + \sum_{i \in S} \sum_{T \in \Omega_{(m+1)}^S} q_i^T = \sum_{T \in \Omega_m^S} q(T) + A_{m+1}^S.$$

Now, suppose that equation (2) holds for $k = m, m+1, \ldots, r-1 < n$. Then we prove that it also holds for $k = r$. By definition of Q_1^* we have that for $q \in Q_1^*$ holds that $q(T) = \sum_{i \in T} q_i^T = \sum_{j \in N \setminus T} q_j^{T \cup j}$, $T \in \Omega$. Hence, it follows that

$$\begin{aligned}
A^S &= \sum_{T \in \Omega_{r-1}^S} q(T) + A_r^S = \sum_{T \in \Omega_{r-1}^S} q(T) + \sum_{i \in S} \sum_{T \in \Omega_{(r)}^S} q_i^T \\
&= \sum_{T \in \Omega_{r-1}^S} \sum_{j \in N \setminus T} q_j^{T \cup j} + \sum_{i \in S} \sum_{T' \in \Omega_r^S} q_i^{T'} + A_{r+1}^S \\
&= \sum_{T' \in \Omega_r^S} \left(\sum_{j \in T' \setminus S} q_j^{T'} + \sum_{i \in S} q_i^{T'} \right) + A_{r+1}^S \\
&= \sum_{T' \in \Omega_r^S} \sum_{i \in T'} q_i^{T'} + A_{r+1}^S = \sum_{T' \in \Omega_r^S} q(T') + A_{r+1}^S.
\end{aligned}$$

Hence, equation (2) is satisfied for $k = r$. Applying this equation for $k = n$ it follows that

$$A^S = \sum_{T \in \Omega_n^S} q(T) + A_{n+1}^S = \sum_{i \in N} q_i^N = 1.$$

So, $A^S = \sum_{i \in S} \sum_{T \in \Omega^S} q_i^T = 1$ for all $q \in Q_1^*$ and $S \subseteq \Omega$ and thus $q \in Q_1$.
Second we prove that $Q_1 \subseteq Q_1^*$. By definition, we have for any $q \in Q_1$ that

$$\sum_{i \in N} q_i^N = \sum_{i \in N} \sum_{T \in \Omega^N} q_i^T = 1.$$

It remains to prove that for any $q \in Q_1$ and $S \in \Omega$, $|S| \leq n-1$, holds

$$q(S) = \sum_{j \in N \setminus S} q_j^{S \cup j} \tag{3}$$

To do so, fix an arbitrary $q \in Q_1$. For $S \in \Omega$ with $|S| = n-1$ and thus $S = N \setminus \{i\}$ for some $i \in N$ we have that

$$\sum_{j \in N \setminus i} (q_j^{N \setminus i} + q_j^N) = q(N \setminus i) + \sum_{j \in N \setminus i} q_j^N = 1.$$

With $\sum_{j \in N} q_j^N = 1$ and $S = N \setminus \{i\}$ it follows that

$$q(S) = 1 - \sum_{j \in N \setminus i} q_j^N = q_i^N = \sum_{j \in N \setminus S} q_j^N,$$

which shows that equation (3) holds when $|S| = n-1$. Now, suppose that equation (3) is valid for any S with $|S| = n-1, n-2, \ldots, n-(r-1) > 1$ and take some S with $|S| = n-r$. Since $q \in Q_1$ it follows that

$$\begin{aligned}
1 &= \sum_{i \in S} \sum_{T \in \Omega^S} q_i^T = q(S) + \sum_{i \in S} \sum_{T \in \Omega^S, T \neq S} q_i^T \\
&= q(S) + \sum_{i \in S} \sum_{\emptyset \neq T \subseteq N \setminus S} q_i^{S \cup T} = q(S) + \sum_{\emptyset \neq T \subseteq N \setminus S} \left(\sum_{i \in S \cup T} q_i^{S \cup T} - \sum_{i \in T} q_i^{S \cup T} \right) \\
&= q(S) + \sum_{k=1}^{r} \sum_{T \in \Omega_S^k} \left(q(S \cup T) - q(T,\ S \cup T) \right) \\
&= q(S) + q(N) + \sum_{k=1}^{r-1} \left[\sum_{T \in \Omega_S^k} q(S \cup T) - \sum_{T \in \Omega_S^{k+1}} q(T, S \cup T) \right] - \sum_{T \in \Omega_S^1} q(T, S \cup T),
\end{aligned}$$

where $q(T,\ S \cup T) = \sum_{i \in T} q_i^{S \cup T}$. By applying the induction hypothesis we have for any $k = 1, \ldots, r-1$ that

$$\sum_{T \in \Omega_S^k} q(S \cup T) = \sum_{T \in \Omega_S^k} \sum_{j \in N \setminus (S \cup T)} q_j^{S \cup T \cup j} = \sum_{T \in \Omega_S^{k+1}} \sum_{j \in T} q_j^{S \cup T} = \sum_{T \in \Omega_S^{k+1}} q(T, S \cup T),$$

so that within each of the square brackets in the previous equation the first term is equal to the second term and thus

$$1 = q(S) + q(N) - \sum_{T \in \Omega_S^1} q(T,\ S \cup T) = q(S) + 1 - \sum_{j \in N \setminus S} q_j^{S \cup T}.$$

This proves that equation (3) holds for any $S \in \Omega$ and thus $q \in Q_1^*$. Q.E.D.

Proof of Lemma 4.3.

From Lemma 4.2 it follows that any $q \in Q^W$ is also in Q_1^* and thus satisfies the relations $q(S) = \sum_{j \in N \setminus S} q_j^{S \cup j}$ for all $S \in \Omega$, $S \neq N$. So, for any $k < n$ we have that

$$\sum_{S \in \Omega_k} q(S) = \sum_{S \in \Omega_k} \sum_{j \in N \setminus S} q_j^{S \cup j} = \sum_{T \in \Omega_{k+1}} \sum_{j \in T} q_j^{(T \setminus j) \cup j} = \sum_{T \in \Omega_{k+1}} q(T)$$

Applying this recursively for $k = 1, \ldots, n-1$ it follows that

$$\sum_{S \in \Omega_k} q(S) = \sum_{T \in \Omega_n} q(T) = q(N) = 1,\ k = 1, \ldots, n.$$

Q.E.D.

Proof of Lemma 4.4.

We first prove the assertions (iii) and (iv). Fix some $\pi \in \Pi$ and $i \in N$. First, consider $S = S_i^\pi$. Then by definition of p^π we have that

$$F_i^S(p^\pi) = \sum_{T \in \Omega^S} (-1)^{|T|-|S|} p_i^{\pi,T} = 1 \geq 0.$$

Next, consider $S \subset S_i^\pi$ with $S_i^\pi \setminus S \equiv R \neq \emptyset$. Then

$$F_i^S(p^\pi) = \sum_{T \in \Omega^S} (-1)^{|T|-|S|} p_i^{\pi,T} = \sum_{T \subseteq R} (-1)^{|T|} p_i^{\pi, S \cup T} = \sum_{T \subseteq R} (-1)^{|T|} = 0,$$

because in the latter sum the number of positive terms equals the number of negative terms. Finally, in case that $S \setminus S_i^\pi \neq \emptyset$ we have by definition that $p_i^{\pi,T} = 0$ for all $T \in \Omega^S$ and thus $F_i^S(p^\pi) = 0$. Summarizing we have shown that for all $i \in N$ and $S \in \Omega$ holds

$$\begin{aligned} F_i^S(p^\pi) &= 1 \text{ if } S = S_i^\pi, \\ &= 0 \text{ otherwise,} \end{aligned}$$

which shows that $F(p^\pi) = q^\pi$, $\pi \in \Pi$, which proves assertion (iii). On the other hand, assertion (iv) follows directly from the definition of q^π by observing that

$$\begin{aligned} G_i^S(q^\pi) = \sum_{T \in \Omega^S} q_i^{\pi,T} = q_i^{\pi, S_i^\pi} &= 1 \text{ if } S \subseteq S_i^\pi, \\ &= 0 \text{ otherwise} \end{aligned}$$

and thus $G(q^\pi) = p^\pi$.

To prove assertions (i) and (ii), first observe that $F(p^\pi) = q^\pi$, so that for all $i \in N$ and $S \subset \Omega^i$ holds

$$F_i^S(p^\pi) = \sum_{T \in \Omega^S} (-1)^{|T|-|S|} p_i^{\pi,T} = q_i^{\pi,S} \geq 0,$$

which shows that $p^\pi \in P^W$. On the other hand, from $G(q^\pi) = p^\pi$ it follows for all $S \in \Omega$ that

$$\sum_{i \in S} \sum_{T \in \Omega^S} q_i^{\pi,T} = \sum_{i \in S} G_i^S(q^\pi) = \sum_{i \in S} p_i^{\pi,S} = 1,$$

which shows that $q^\pi \in Q^W$. To prove that p^π and q^π are extreme points of the compact, convex polyhedrons P^W and Q^W respectively, it follows from the nonnegativity of the elements of P^W and Q^W and the equalities $\sum_{i \in S} p_i^S = 1$ and $\sum_{S \in \Omega^i} q_i^S$ that for any $i \in N$ and $S \in \Omega^i$

$$0 \leq p_i^S \leq 1, \text{ for any } p \in P^W$$

and

$$0 \leq q_i^S \leq 1, \text{ for any } q \in Q^W.$$

Now, suppose that for some $\pi \in \Pi$ we have that $p^\pi = \frac{1}{2}(p + \widehat{p})$ for some $p,\ \widehat{p} \in P^W$. Since $p_i^{\pi,S} \in \{0, 1\}$ it follows from the inequalities above that we must have that

$$p_i^{\pi,S} = p_i^S = \widehat{p}_i^S, \text{ for all } i \in N,\ S \in \Omega^i$$

and thus $p^\pi = p = \widehat{p}$, which proves that $p^\pi \in \text{Ex}(P^W)$. Analogously it follows that $q^\pi \in \text{Ex}(Q^W)$. Q.E.D.

Proof of Proposition 4.5.
For $n = 1$, the lemma is true by definition. So, suppose $n \geq 2$. Because of (i) of Lemma 4.4 we only have to prove that $\text{Ex}(Q^W) \subseteq \{q^\pi | \pi \in \Pi\}$. Let q be an extreme point of Q^W. We first prove that for any $S \in \Omega$ holds that $q_i^S \in \{0, 1\}$ for all $i \in S$. To do so, let m, $0 \leq m \leq n$, be the largest number such that for any S with $|S| \leq m$ holds that $q_i^S \in \{0, 1\}$ for all $i \in S$ and thus also $q(S) \in \{0, 1\}$ because of Lemma 4.3. When $m = n$, q satisfies the requirements. It is clear, that in case $m \neq n$ we have $m < n-1$, since otherwise, when $m = n-1$, we have for all $j \in N$ that $q(N \setminus j) \in \{0, 1\}$. Hence it follows from $q(N \setminus j) = q_j^N$ that $q_j^N \in \{0, 1\}$ and hence $q_i^S \in \{0, 1\}$ for all $i \in S$ and $S \in \Omega$, which contradicts to the equality $m = n - 1$.

Now, suppose, $0 \leq m < n-1$, i.e. there exists S_1 and $j_1 \in S_1$ with $|S_1| = m+1$ and $0 < q_{j_1}^{S_1} < 1$. Then we prove that q is not an extreme point of Q^W by constructing a nonzero solution $b \in Q$ of the homogeneous system

$$b(N) = 0 \text{ and } b(S) = \sum_{j \in N \setminus S} b_j^{S \cup j}, \; S \in \Omega, S \neq N. \tag{4}$$

When $m = 0$, and thus $S_1 = \{j_1\}$, there exists an index $k_1 \neq j_1$ such that $0 < q_{k_1}^{T_1} < 1$, where $T_1 = \{k_1\}$, because according to Lemma 4.3 $\sum_{S \in \Omega_1} q(S) = \sum_{i \in N} q_i^{\{i\}} = 1$. When $m \geq 1$, take $S = S_1 \setminus \{j_1\}$. Since $|S| = m$ and thus $q(S) \in \{0, 1\}$, it follows from $q(S) = \sum_{j \in N \setminus S} q_j^{S \cup j} \geq q_{j_1}^{S_1} > 0$ that $q(S) = 1$ and thus there exists at least one other element $k_1 \in N \setminus S$ such that $q_{k_1}^{T_1} > 0$, where $T_1 = S \cup \{k_1\}$. In both cases ($m = 0$ and $m \geq 1$), we have that both $q(S_1) > 0$ and $q(T_1) > 0$ and thus it follows from $q(R) = \sum_{j \in N \setminus R} q_j^{R \cup j}$ for all $R \in \Omega \setminus \{N\}$ that there exists $j_2 \in N \setminus S_1$ and $k_2 \in N \setminus T_1$ such that $q_{j_2}^{S_1 \cup j_2} > 0$, respectively $q_{k_2}^{T_1 \cup k_2} > 0$. Now, proceeding the way j_2 and k_2 were constructed, we get two sequences $j_1, j_2, \ldots, j_{n-m}$ and $k_1, k_2, \ldots, k_{n-m}$ in $N \setminus S$ satisfying for $r = 0, \ldots, n-m-1$,

$$q_{j_{r+1}}^{S_{r+1}} > 0,$$

$$q_{k_{r+1}}^{T_{r+1}} > 0,$$

where $S_0 = T_0 = S$, $S_{r+1} = S_r \cup \{j_{r+1}\}$ and $T_{r+1} = T_r \cup \{k_{r+1}\}$, $r = 0, \ldots, n-m-1$ (with $S_0 = T_0 = \emptyset$ in case $m = 0$). Since by definition $S_1 \neq T_1$, the sets S_r and T_r are strictly increasing in r and $S_{n-m} = T_{n-m} = N$, there exists an $\ell^* \leq n-m$ such that

$$S_r \neq T_r, \; r = 1, \ldots \ell^* - 1 \text{ and } S_{\ell^*} = T_{\ell^*}.$$

Let

$$q_{\min} = \min\{q_{j_r}^{S_r}, \; q_{k_r}^{T_r}, \; r = 1, \ldots, \ell^*\}.$$

By construction we have that $q_{\min} > 0$. Define $b \in Q$ satisfying the conditions in (4) by

$$\begin{aligned} b_i^R &= q_{\min} \text{ if } R = S_r, \; i = j_r, \; r = 1, \ldots, \ell^*, \\ &= -q_{\min} \text{ if } R = T_r, \; i = k_r, \; r = 1, \ldots, \ell^*, \\ &= 0 \quad \text{ otherwise.} \end{aligned}$$

Because b satisfies the conditions of (4), both $q + b$ and $q - b$ are in Q_1^*. Moreover, by definition, all components of $q+b$ and $q-b$ are nonnegative and thus both weights systems belong to Q^W, while $q = \frac{1}{2}((q+b)+(q-b))$. Hence, q is not an extreme point of Q^W. This contradiction proves that $q_i^T \in \{0, 1\}$ for any $T \in \Omega$ and any $i \in T$ when q is an extreme point of Q^W.

It remains to show that any extreme point belongs to $\{q^\pi | \pi \in \Pi\}$. Since by definition $\sum_{i \in S} \sum_{T \in \Omega^S} q_i^T = 1$ for any $S \in \Omega$, we have by taking $S = N$ that $\sum_{i \in N} q_i^N = 1$. Since $q_i^N \in \{0, 1\}$ for all $i \in N$, there exists a unique index $i_n \in N$, such that $q_{i_n}^N = 1$ and $q_j^N = 0$ for all $j \neq i_n$. Since $Q^W \subseteq Q_1$, it follows from Lemma 4.2 that the equality relations in Q_1^* hold. Taking some $j \in N$ and $S = N \setminus \{j\}$ we have $q(N \setminus \{j\}) = q_j^N$ and thus

$$\begin{aligned} q(N \setminus j) &= 1 \text{ if } j = i_n, \\ &= 0 \text{ if } j \neq i_n. \end{aligned}$$

Hence, there exists a unique index i_{n-1} such that $q_{i_{n-1}}^{N \setminus i_n} = 1$ and $q_j^{N \setminus i_n} = 0$ for all $j \notin \{i_n, i_{n-1}\}$. In general, for given k, $2 \leq k \leq n$, let $i_k, \ldots, i_n$ be a collection of indices in N and such that for $r = k, \ldots, n$,

$$\begin{aligned} q_i^{S_{r+1}} &= 1 \text{ if } i = i_r, \\ &= 0, \text{ if } i \in S_r, \end{aligned}$$

where $S_r = N \setminus \{i_r, \ldots, i_n\}$ (with $S_{n+1} \equiv N$). For S_k it follows from applying the equality (holding for any $q \in Q_1^* = Q_1$)

$$q(S_k) = \sum_{j \in N \setminus S_k} q_j^{S_k \cup j} = q_{i_k}^{S_{k+1}} = 1.$$

Applying that $q_i^S \in \{0, 1\}$ for all $i \in S \in \Omega$ it follows that there exists a unique index $i_{k-1} \in S_k$ such that

$$\begin{aligned} q_i^{S_k} &= 1 \text{ if } i = i_{k-1}, \\ &= 0, \text{ if } i \in S_{k-1}, \end{aligned}$$

where $S_{k-1} = S_k \setminus i_{k-1}$. So, applying this procedure subsequently for $k = n, n-1, \ldots, 2$, we get a uniquely determined sequence $i_n, i_{n-1}, \ldots, i_1$ such that $q = q^\pi$ with $\pi = (i_1, \ldots, i_n) \in \Pi$. This shows that q is in $\{q^\pi | \pi \in \Pi\}$. Q.E.D.

Proof of Lemma 4.6.

(i) To prove that G is a nondegenerate linear operator we have to show that $G(q) = 0$ implies that $q = 0$. To do so, suppose $G(q) = 0$. Then $G_i^N(q) = q_i^N = 0$ for any $i \in N$. We proceed by induction and suppose that $G(q) = 0$ implies that $q_i^S = 0$ for any $i \in N$ and any $S \in \Omega_k^i$ for $r + 1 \leq k \leq n$. Consider $S \in \Omega_r^i$. Then

$$0 = G_i^S(q) = q_i^S + \sum_{T \in \Omega^S, T \neq S} q_i^T = q_i^S$$

and thus $q_i^S = 0$ for any $i \in N$ and S with $|S| = r$. It follows by the induction hypothesis that $q_i^S = 0$ for all $i \in N$ and any $S \in \Omega^i$.

(ii) To prove that $F = G^{-1}$ we have to show that $q = F(p)$ is the solution of $G(q) = p$. So, consider the system of linear equations

$$\sum_{T\in\Omega^S} q_i^T = p_i^S,\ S \in \Omega^i,\ i \in N.$$

Clearly, for $S = N$ we obtain that

$$q_i^N = p_i^N = F_i^N(p),\ i \in N.$$

Again we proceed by induction and suppose that for $r+1 \leq k \leq n$ holds

$$q_i^S = F_i^S(p),\ i \in N, S \in \Omega_k^i.$$

Consider $S \in \Omega_r^i$. Then it follows from $p_i^S = G_i^S(q)$ that

$$q_i^S = p_i^S - \sum_{T\in\Omega^S, T\neq S} q_i^T$$

and thus by applying the induction hypothesis that

$$q_i^S = p_i^S - \sum_{T\in\Omega^S, T\neq S} \sum_{T'\subseteq N\setminus T} (-1)^{|T'|} p_i^{T\cup T'}.$$

By observing that $T \cup T' = S \cup [(T \setminus S) \cup T']$ and denoting $S' = (T \setminus S) \cup T'$ the latter expression can be rewritten as

$$q_i^S = p_i^S - \sum_{S'\subseteq N\setminus S, S'\neq\emptyset} p_i^{S\cup S'} \left(\sum_{T'\subseteq S', T'\neq S'} (-1)^{|T'|} \right) = p_i^S + \sum_{S'\subseteq N\setminus S, S'\neq\emptyset} (-1)^{|S'|} p_i^{S\cup S'} = F_i^S(p).$$

(iii) To prove that $G(Q^W) = P^W$ we first show that $G(q) \in P^W$ for all $q \in Q^W$. Let $q \in Q^W$. Since $q \geq 0$, it follows by definition of G that $G(q) \geq 0$. Moreover,

$$\sum_{T\in\Omega^S} (-1)^{|T|-|S|} G_i^T(q) = F_i^S(G(q)) = q_i^S$$

and thus $\sum_{T\in\Omega^S} (-1)^{|T|-|S|} G(q)_i^T \geq 0$. Finally,

$$\sum_{i\in S} G_i^S(q) = \sum_{i\in S} \sum_{T\in\Omega^S} q_i^T = 1.$$

Hence $G(q) \in P^W$. Second, we show that for every $p \in P^W$, there exists $q \in Q^W$ such that $p = G(q)$. For some $p \in P^W$, let $q = F(p)$. Then $G(q) = p$ because $G(q) = G(F(p)) = G \cdot G^{-1}(p) = p$. To prove that $q \in Q^W$, observe that $q = F(p) \geq 0$. Further, for any $S \in \Omega$,

$$\sum_{i\in S} \sum_{T\in\Omega^S} q_i^T = \sum_{i\in S} G_i^S(F(p)) = \sum_{i\in S} p_i^S = 1.$$

Hence $q \in Q^W$. Q.E.D.

Proof of Lemma 4.7.
From the definition of p^π it follows that

$$\begin{aligned}\phi_i^{p^\pi}(v) &= \sum_{S \in \Omega^i} p_i^{\pi,S} \Delta^T(v) = \sum_{S \in \Omega^i | S \subseteq S_i^\pi} \Delta^S(v) \\ &= v(S_i^\pi) - v(S_i^\pi \setminus \{i\}) = m_i^\pi(v).\end{aligned}$$

Q.E.D.

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